QUESTION OF SOLVING NONLINEAR PROBLEMS UNDER THE COMBINED ACTION OF HEAT CONDUCTION AND RADIATION

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The radiant heat transfer of an annular plate to the environment is examined. A simple analytic solution is obtained that describes the stationary temperature field.

Heat transfer is accomplished in a number of cases by heat conduction and radiation. In this connection, the systems of equations describing the temperature fields in solids are nonlinear in the general case and have no exact analytic solution up to now. However, in some cases, particularly for a stationary temperature distribution, a sufficiently simple analytic solution can be obtained that possesses arbitrarily high accuracy. Let us examine the example presented in [1] as an illustration of such a problem. A thin plate in the shape of a ring with the inner and outer radii r_i and r_o is in a vacuum. One of the ring surfaces and the outer edge are heat insulated, while the inner edge is maintained at the constant temperature T_i. The uninsulated ring surface which emits energy into the environment at the temperature $T_e = 0$ is diffusion-gray and has the emissivity ε. Assuming the disc sufficiently thin, constancy of the temperature over the plate thickness b can be allowed. All the data presented are shown in the figure.

Under the heat-transfer conditions described and for the constants ε and λ_o, the question

$$\frac{d^2T}{dr^2}+\frac{1}{r}\frac{dT}{dr}-\frac{\varepsilon\sigma}{\lambda_0 b}T^4=0 \tag{1}$$

is obtained in [1] with the boundary conditions

$$T|_{r=r_i}=T_i, \tag{2}$$

$$\left.\frac{dT}{dr}\right|_{r=r_0}=0. \tag{3}$$

The term with T to the fourth power in (1) makes it nonlinear; consequently, different numerical solution methods are often used to determine the desired temperature from (1)-(3).

One of the possible methods of obtaining an analytic solution is proposed in this paper. To do this, we represent system (1)-(3) in the form

$$\frac{d^2\Theta}{dR^2}+\frac{1}{R}\frac{d\Theta}{dR}-B_i^*(4\Theta-6\Theta^2+4\Theta^3-\Theta^4)=-B_i^*, \tag{4}$$

$$\Theta|_{R=l}=0, \tag{5}$$

$$\left.\frac{d\Theta}{dR}\right|_{R=1}=0 \tag{6}$$

by setting $\Theta=(T_i-T)/T_i$, $R=r/r_o$, $l=r_i/r_o$, $B_i^*=(\varepsilon\sigma T_i^3 r_o^2)/\lambda_o b$. The functions Θ^i (i = 1, 2, 3, 4) in (4) satisfy the Dirichlet conditions [2]. Hence, they can be represented as Fourier series in a certain interval $(0, \Theta_p)$

$$\Theta^i=\sum_{k=1}^{\infty}\alpha_k^i\sin(k\pi\Theta/\Theta_p), \tag{7}$$

where the coefficients α_k^i are determined by the formula

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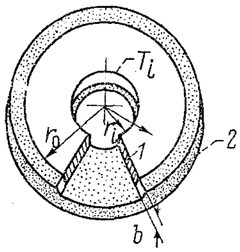

Fig. 1. Thin annular plate, heat insulated on one face and the outer edge: 1) annular rib; 2) insulation.

$$\alpha_k^i = \frac{2}{\Theta_p}\int_0^{\Theta_p} \Theta^i \sin(k\pi\Theta/\Theta_p)\, d\Theta. \tag{8}$$

Substituting (7) into (4)-(6) and applying the reduction rule [3], we obtain

$$\frac{d^2T_k}{dR^2} + \frac{1}{R}\frac{dT_k}{dR} - N_k T_k = -M_k, \tag{9}$$

$$T_k|_{R=l} = 0, \tag{10}$$

$$\left.\frac{dT_k}{dR}\right|_{R=1} = 0, \tag{11}$$

where $T_k = \alpha_k^1 \sin(k\pi\Theta/\Theta_p)$; $M_k = B_i^*/(e(k-1)!)$; $N_k = (4\alpha_k^1 - 6\alpha_k^2 + 4\alpha_k^3 - \alpha_k^4)\alpha_k^1$; and e is the base of natural logarithms. According to [4], the general solution of (9) has the form

$$T_k = \frac{M_k}{N_k} + C_1 I_0(\sqrt{N_k}R) + C_2 K_0(\sqrt{N_k}R). \tag{12}$$

Taking account of conditions (10) and (11), we represent the solution (12) as:

$$T_k = \frac{M_k}{N_k}\left[1 - \frac{I_0(\sqrt{N_k}R)K_1(\sqrt{N_k}) + I_1(\sqrt{N_k})K_0(\sqrt{N_k}R)}{I_0(\sqrt{N_k}l)K_1(\sqrt{N_k}) + I_1(\sqrt{N_k})K_0(\sqrt{N_k}l)}\right]. \tag{13}$$

Summing (13) over k, we find the desired solution of system (4)-(6), i.e.,

$$\Theta(R) = \sum_{k=1}^{\infty}\frac{M_k}{N_k}\left[1 - \frac{I_0(\sqrt{N_k}R)K_1(\sqrt{N_k}) + I_1(\sqrt{N_k})K_0(\sqrt{N_k}R)}{I_0(\sqrt{N_k}l)K_1(\sqrt{N_k}) + I_1(\sqrt{N_k})K_0(\sqrt{N_k}l)}\right]. \tag{14}$$

Since the quantity Θ_p is still unknown, then N_k in solution (14) has also not been determined. To find Θ_p, we set R = 1 in (4), then

$$\left.\frac{d^2\Theta}{dR^2}\right|_{R=1} + B_i^*(1 - \Theta|_{R=1})^4 = 0. \tag{15}$$

Taking into account that

$$\left.\frac{d^2\Theta}{dR^2}\right|_{R=1} = -\sum_{k=1}^{\infty}\frac{M_k}{\sqrt{N_k}}\frac{1}{I_0(\sqrt{N_k}l)K_1(\sqrt{N_k}) + I_1(\sqrt{N_k})K_0(\sqrt{N_k}l)}$$

or

$$\Theta(R)|_{R=1} = \sum_{k=1}^{\infty}\frac{M_k}{N_k}\left[1 - \frac{1}{\sqrt{N_k}}\frac{1}{I_0(\sqrt{N_k}l)K_1(\sqrt{N_k}) + I_1(\sqrt{N_k})K_0(\sqrt{N_k}l)}\right],$$

then Θ_p can be found from (15). The procedure for finding the root Θ_p reduces to solving a transcendental equation. From the equation

$$N_k = B_i^*\left[4 - 6\Theta_p\left(1 - \frac{2}{(k\pi)^2} + \frac{2}{(k\pi)^2\cos k\pi}\right) + 4\Theta_p^2\left(1 - \frac{6}{(k\pi)^2}\right) - \Theta_p^3\left(1 - \frac{12}{(k\pi)^2} + \frac{24}{(k\pi)^4}\left(1 - \frac{1}{\cos k\pi}\right)\right)\right]$$

TABLE 1. Results of an Analytic and Numerical Computation of the Temperature Field in an Annular Plate for l = 0.1

B_i^*	R				
	0,2	0,4	0,6	0,8	1,0
0,2	0,04451 0,04423	0,08513 0,08477	0,1041 0,1039	0,1137 0,1135	0,1164 0,1161
20,0	0,3957 0,3913	0,6296 0,6269	0,6875 0,6851	0,7043 0,7035	0,7071 0,7065

TABLE 2. Results of Analytic and Numerical Computations of the Temperature Field in an Annular Plate for l = 0.9

B_i^*	R					
	0,9	0,92	0,94	0,96	0,98	1,0
0,2	0,0 0,0	0,3772E-3 0,3686E-3	0,6681 E-3 0,6679 E-3	0,8728 E-3 0,8722 E-3	0,9944 E-3 0,9935 E-3	0,1034 E-2 0,1033 E-2
20,0	0,0 0,0	0,0294 0,02907	0,052 0,05194	0,0673 0.06713	0,07612 0,07598	0,0791 0,07887

all the N_k needed to define the solution (14) completely are found from the value found for Θ_p. Convergence of the series (14) is indubitable ($M_k = B_i^*/(e(k-1)!)$) and the quantity of terms in the series is determined in each specific case by the required accuracy of the calculations.

Series (14) evidently satisfies conditions (6) and (7) and Eq. (15). Conformity of solution (14) to (4) requires proof. Since it is still not possible to investigate this question analytically, we use a numerical solution of system (4)-(6). Data on the solution (14) (upper row) and numerical integration by finite differences for the second-order nonlinear equations (lower row) are presented in Tables 1 and 2 for different values of l and B_i^*.

It follows from an analysis of the tables that the solution (14) obtained possesses high accuracy and satisfies both the equation itself and the boundary conditions.

Therefore, it follows from the above that for such a class of problems it is always possible to obtain a solution in analytically simple form as most convenient in the sense of analysis of the thermal phenomena and computation.

NOTATION

T, running temperature; T_i, temperature of the inner edge of the ring; r, running radius; ε, emissivity; σ, Stefan—Boltzmann constant; λ_0, heat conductivity of the material; b, thickness of the annular plate; Θ, relative temperature; and $R = r/r_0$, relative running radius.

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